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Isolated inertialess drops cannot break up

By JENS EGGERS[1] AND MARCO A. FONTELOS[2]†

[1]School of Mathematics, University of Bristol, University Walk, Bristol BS8 1TW, UK

[2]Departamento de Matemática Aplicada, Universidad Rey Juan Carlos, C/Tulipán S/N, 28933 Móstoles, Madrid, Spain

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We consider an axisymmetric, freely suspended fluid drop with surface tension, whose viscosity is so large that both inertia and forcing by an external fluid can be ignored. We show that whatever the initial condition, pinchoff can never occur.

The breakup of fluid drops has been studied very extensively (Stone 1994; Eggers 1997) owing to its relevance to mixing (Cristini & Tan 2004), printing (Basaran 2002) and DNA analysis (Basaran 2002; Schena *et al.* 1998). In most circumstances, breakup of an extended piece of fluid occurs almost inevitably owing to the Rayleigh instability (Eggers 1997), which tends to locally reduce the radius until it reaches zero. So one might think that a sufficiently extended drop, that has separated from a nozzle or has been stretched by an external fluid of comparatively low viscosity, will break up in the same manner. Experiments and numerical simulations however indicate that this is not the case, as long as inertia can be neglected. Stone (1994), p. 81, observes that experiments and numerical simulations (Stone & Leal 1989) suggest that the maximum drop length needed to ensure breakup increases roughly linearly with the viscosity ratio between the drop and the surrounding fluid. Here we treat the limit that the outer fluid can be neglected altogether, in which case we can show that the drop never breaks up, but rather retracts to its spherical state of minimum surface energy, see figure 1. Qualitatively, this means that the ends of the drop retract before the drop can break in the middle. Inertia or an external fluid inhibit retraction, so pinching may occurs if either effect is taken into account and the drop is sufficiently extended.

Neglecting inertia, the interior of the drop is described by Stokes' equation, subject to a normal stress $\gamma\kappa\boldsymbol{n}$, where γ is the coefficient of surface tension and $\kappa = 1/h(1+h_z^2)^{1/2} - h_{zz}/(1+h_z^2)^{3/2}$ is twice the mean curvature of the interface. If $\boldsymbol{\sigma}$ is the stress tensor, this can be summarized concisely by

$$\nabla\cdot\boldsymbol{\sigma} = 0 \quad \text{in the drop,} \quad \boldsymbol{\sigma}\cdot\boldsymbol{n} = -\gamma\boldsymbol{n}\kappa \quad \text{on the surface.} \tag{1}$$

Integrating $\nabla\cdot\boldsymbol{\sigma}$ over a volume V bounded by the drop surface and a plane perpendicular to the axis (cf. figure 1), we find from the divergence theorem and from the boundary condition that

$$0 = \int_S \boldsymbol{n}\cdot\boldsymbol{\sigma}\,\mathrm{d}s = \int_{Cr(z)} \boldsymbol{n}\cdot\boldsymbol{\sigma}\,\mathrm{d}s + \int_O \boldsymbol{n}\cdot\boldsymbol{\sigma}\,\mathrm{d}s = \int_{Cr(z)} \boldsymbol{e}_z\cdot\boldsymbol{\sigma}\,\mathrm{d}s - \gamma\int_O \boldsymbol{n}\kappa\,\mathrm{d}s, \tag{2}$$

where O is the surface as shown in figure 1, and $Cr(z)$ is the cross-section of the drop at z.

† Present address: Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain.

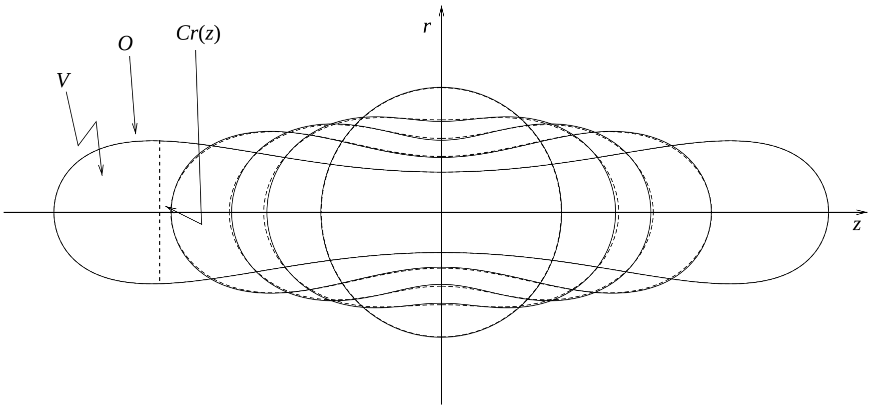

FIGURE 1. A viscous drop of unperturbed radius R is initially extended to a length of $15.5R$. The solid lines show it retracting back into a sphere following Stokes' equation, at $t = n \times 0.646\eta R/\gamma$, $n = 0, 1, 2, 3$. The dashed lines are profiles computed on the basis of the long-wave equation (7), shown at the same times. Note that the minimum local drop radius is always a monotonically increasing function.

Using $\boldsymbol{n} = (-h_z\boldsymbol{e}_z + \boldsymbol{e}_r)/(1 + h_z^2)^{1/2}$, the integral over O can be evaluated as

$$-2\pi\gamma\boldsymbol{e}_z \int_{z_{end}}^{z} h h_z \kappa \, \mathrm{d}z = -2\pi\gamma\boldsymbol{e}_z \int_{z_{end}}^{z} \left[h/\left(1 + h_z^2\right)^{1/2}\right]_z \mathrm{d}z = -2\pi\gamma\boldsymbol{e}_z h/\left(1 + h_z^2\right)^{1/2}, \quad (3)$$

since the height $h(z, t)$ goes to zero at the end of the drop. Thus we arrive at the exact relation

$$\int_0^{h(z,t)} \boldsymbol{e}_z \cdot \boldsymbol{\sigma} r \, \mathrm{d}r = -\gamma\boldsymbol{e}_z h/\left(1 + h_z^2\right)^{1/2} \quad (4)$$

for the total force on the cross-section of the drop.

To evaluate the z-component of (4), we note that $\boldsymbol{e}_z \cdot \boldsymbol{\sigma} \cdot \boldsymbol{e}_z = -p + 2\eta\partial v_z/\partial z = -p - (2\eta/r)\partial(r v_r)/\partial r$, where $p(r, z)$ is the pressure, η the viscosity of the liquid, and $v_z(r, z)$, $v_r(r, z)$ are the axial and radial components of the velocity, respectively. Thus the z-component of (4) can be rewritten as

$$\int_0^{h(z)} p(r, z) r \, \mathrm{d}r + 2\eta h(z) v_r(h(z), z) = \gamma h/\left(1 + h_z^2\right)^{1/2}. \quad (5)$$

We are not able to evaluate the integral over the pressure exactly, so to make further progress we consider the expansion of the pressure in r. Following Eggers (1997, p. 887), this gives

$$p(r, z) = \gamma\kappa + 2\eta v_r(h, z)/h + O\left(r^2, r h_z, h_z^2\right). \quad (6)$$

Neglecting the higher-order terms in (6), (5) finally becomes

$$v_r(h(z), z) = (\gamma/6\eta)\left(1/\left(1 + h_z^2\right)^{1/2} + h h_{zz}/\left(1 + h_z^2\right)^{3/2}\right). \quad (7)$$

At a local minimum of h, h_{zz} is positive, making v_r positive, so h_{min} is *increasing* in time and breakup is impossible.

We have been careful to invoke (7) only at the point h_{min}, where h_z is zero. If breakup were to occur, h_{min} would have to go to zero, and higher-order radial terms in (6) can be neglected. In fact, expansions in the slenderness like (6) have been used successfully to describe the pinch-off of liquid drops for Navier–Stokes (Eggers 1993) and Stokes (Papageorgiou 1995) dynamics. Local analysis shows that h_{min} goes to zero much faster than a typical axial scale, so higher-order radial terms go to zero as the singularity is approached. In the present case, of course, a singularity never occurs. This does not contradict previous work on breakup, since all existing analyses of breakup are local in character, and thus assume from the outset that h_{min} goes to zero somewhere.

As indicated above, (7) is equivalent to the one-dimensional long-wave-type description of liquid filaments, which was found to often work surprisingly well throughout a fluid drop or filament (Eggers 1997; Basaran 2002). Here, we find this observation confirmed, as illustrated in figure 1 by superimposing the long-wave calculation (dashed lines) onto the full numerical calculation. No adjustable parameter was introduced in the comparison. It follows from (7) and is illustrated by figure 1, that the minimum radius as given by the long-wave approximation must be monotonically increasing. We suspect that the same holds true for the full Stokes equations, but at present we cannot exclude a decreasing h_{min} in cases where h_{min} is not small. We reiterate that although (7) works very well in practice in describing entire drops, we are only required to make use of it locally to show the absence of breakup. Our argument also does not make use of axisymmetry in any essential way, but an analysis of this more general situation is outside the scope of this short note.

It would be interesting to extend our results to the case that an exterior fluid of small but finite viscosity is present. Lister & Stone (1998) point out that the local pinch-off behaviour is modified if shear stresses in the exterior fluid become comparable to those in the interior of the bubble. If the exterior viscosity is small, this only happens if h_{min} is correspondingly small. Since we have demonstrated in this note that an isolated bubble never comes close to pinch-off, the modifications in the pinching behaviour introduced by an external fluid (Cohen *et al.* 1999; Sierou & Lister 2003) never become relevant. What has not been explored is the effect of the outer fluid on the retraction dynamics, which may become relevant if the drop is highly extended. A possible way to approach this problem is through a scaling analysis similar to that of Powers *et al.* (1998), who investigated the opposite limit of small drop viscosity.

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